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Roughness of (μ_1,μ_2) – Dichotomies of first order Sylvester matrix Differential systems in L∞

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Abstract

Main objective of this paper is to present the concept of ($\mu 1,\mu 2$) – dichotomy. To be precise, we present some conditions for (μ_1,μ_2) – dichotomy under small perturbations associated with Kronecker product Sylvester Matrix differential homogeneous System,
X’ (t) =A(t)X(t)+X(t)B(t) in L∞.

Keywords: Dichotomy, Kronecker product, Matrix Sylvester differential systems..

Introduction

In this paper we focus our attention to study on (μ_1,μ_2) – dichotomy for the first order Matrix Sylvester differential

$$X' = A(t)X + XB(t) \qquad (1.1)$$

Where A(t) and B(t) are matrices of order n×n, tєJ, where $J=[t_0,t_1]$, $-\infty < t_0 \le t_1 < \infty$. The authors Murty, Kumar and Rao{[1],[5] [6]} were developed the results Ψ-dichotomies of linear dynamic system of the form (1.1). Here we provide the more interesting and more challenging result (μ_1,μ_2) – dichotomy under small perturbations associated with Kronecker product Sylvester Matrix differential homogeneous System,

Preliminaries

In this section we give a short over view on some basic results on Kronecker product techniques that are important for the present treatment of (μ_1,μ_2) – dichotomy.

Definition. [4] If P, Q∈ $C^{n\times n}$ are two square matrices of order ‘n’ then their Kronecker product(or direct product or tensor product) is denoted by $P \otimes Q \in C^{n^2\times n^2}$ is defined to be partition matrix

$$P\otimes Q=\begin{bmatrix} p_{11}Q & p_{12}Q & \cdots & p_{1n}Q \\ p_{21}Q & p_{22}Q & \cdots & p_{2n}Q \\ \vdots & \vdots & \cdots & \vdots \\ p_{n1}Q & p_{n2}Q & \cdots & p_{nn}Q \end{bmatrix}$$

We shall make use of vector valued function denoted by Vec P of a matrix $P=\{p_{ij}\}\in C^{n\times n}$ defined by

$$\hat{P}=VecP=\begin{bmatrix} P_{.1} \\ P_{.2} \\ \vdots \\ \vdots \\ P_{.n}\end{bmatrix} \text{ where } P_{.j}=\begin{bmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ \vdots \\ p_{nj}\end{bmatrix} \quad 1\le j\le n$$

it is clear that VecP is of order n^2.

The Kronecker product has the following properties[4]

1. $(P\otimes Q)^* = P^*\otimes Q^*$ (P^* denotes the transpose of P)
2. $(P\otimes Q)^{-1} = P^{-1}\otimes Q^{-1}$
3. The mixed product rule($(P\otimes Q)((M\otimes N) = (PM\otimes QN)$ .This rule holds good, provided the dimension of the matrices are such that the various expressions exist.
4. If P(t) and Q(t) are matrices, then $(P\otimes Q)' = P'\otimes Q'$ ($' = d/dt$)
5. $Vec(PYQ) = (Q^*\otimes P)Vec\,Y$
6. If P and Q are matrices both of order n×n then
   (i) $Vec(PX) = (I_n\otimes P)VecX$
   (ii) $Vec(XP) = (P^*\otimes I_n)VecX$.

Now we apply the vec operator to the system (1.1) and using the kronecker product techniques, then the corresponding kroenecker product system or Sylvester Matrix differential system associated with (1.1) is

$\hat{X}'(t) = [(B^T \otimes I_n) + (I_n \otimes A)]\hat{X}(t)$
(2.1)
$\Rightarrow \phi^1 = G\phi$
(2.2)
where $G = [(B^T \otimes I_n) + (I_n \otimes A)],$ and $\phi = \hat{X}$, G(t) is a continuous matrix on $J \subset R^+$

***Lemma*[1]** Let Y(t) be the fundamental matrix of X'(t) = A(t) X(t) and W(t) is a fundamental matrix of X' = B^T(t)X(t), then (W$\otimes$Y)(t) is the fundamental matrix of (2.2).

Definition. If μ_1 and μ_2 are continuous real valued functions on J, the system $\phi'(t) = G(t)\phi(t)$, will be said to have an (μ_1,μ_2) – dichotomy if there exists supplementary projections P_1 and P_2 { where $P_1 = R_1 \otimes Q_1$, $P_2 = R_2 \otimes Q_2$, R_1, R_2 and Q_1, Q_2 are projections of fundamental matrices Y(t) and W(t) respectively}

on R^{n^2} and a constant L > 0 such that

$$\left\|(W \otimes Y)(t)P_i(W \otimes Y)^{-1}\right\| \le \exp\int_s^t \mu_i(T)dT \quad \text{if}$$

$(t-s)(-1)^{i-1} \ge 0$, i= 1,2 (2.3)

In the case that μ_1 and μ_2 are constants, the system (2.2) is said to have an exponential dichotomy if $\mu_1 < 0 < \mu_2$ and an ordinary dichotomy if $\mu_1 = \mu_2 = 0$.

The condition (2.3) is readily seen to be equivalent to

$$\|(W \otimes Y)(t)P_i\psi\| \le \exp\int_s^t \mu_i(T)dT\|(\psi \otimes Y)(s)P_i\psi\|, \quad \text{if } (t-s)(-1)^{(i-1)} \ge 0, i = 1,2 \qquad (2.4)$$

$$\left\|(W \otimes Y)(t)P_i(W \otimes Y)^{-1}\right\| \le L, \quad i = 1,2. \qquad (2.5)$$

Condition (2.5) says that the angle between $(W \otimes Y)P_i R^{n^2}$ (i=1,2) remains bounded away from zero .The interesting cases are where J is R_+ and R. Continuing now onwards we suppose that J=R.

J.S. Muldowney in [3] had introduced this definition and three sets of necessary and sufficient conditions for a(μ_1 ,μ_2)- dichotomy in terms of Lyapunov function are given. A practical critetria for dichotomy is given in each result, including the extension to unbounded matrices of criteria which depend on a concept of diagonal dominance for *G*(*t*). An asymptotic analysis also given for subspaces of the solution set by means of the associated compound equations.

This paper is devoted to further investigations of this concept of dichotomy. More precisely we prove that, under suitable conditions on (μ_1 ,μ_2),this type of dichotomy has the roughness property with respect to small perturbations in L^∞.

Roughness of a (μ_1, μ_2) – dichotomy

In the proof of the roughness property of a (μ_1, μ_2)-dichotomy, it will be very important that the following "functions":

$$u^*(t) = \int_{-\infty}^{t} \exp\left(\int_s^t \{I \otimes [\mu_1(T) - \mu_2(T)]\}dT\right)dS$$

(3.1)

$$v^*(t) = \int_t^{\infty} \exp\left(\int_s^t \{I \otimes [\mu_2(T) - \mu_1(T)]\}dT\right)dS$$

(3.2)

be well defined and bounded on R^{n^2} .We sketch a few facts that will help us to clarify this point.

Throughout the following ,let

M= $\left\{(I \otimes F) : R \to C^{n^2}\right\}$ (I$\otimes$F) continuous and bounded

Let I$\otimes$F = f, and for any 'f' in M,

$$|f| = Sup\{|f(t)| : t \in R\}$$

The subset N of M denotes the set of almost periodic functions. The subset P_w of 'N' denotes the set of periodic functions of period 'w'. A n^2xn^2 matrix function of R is said to belong to one of these spaces if each column belongs to the space.

Let us define the operator

Zφ = φ^1 – G φ with G$\in$ M. The operator Z is said to be 'regular' if the equation

Zφ = (I$\otimes$F) = f, has a unique solution in 'M' for any f$\in$ M, If the equation Zφ = f has at least one solution in M, for each f$\in$ M, Z is said to be 'weakly regular'. In general, weak regularity does not imply regularity.

***Lemma*:** Suppose that $\alpha \in$ N and

S(α) = $\lim_{s \to \infty} \frac{1}{2s}\int_{-s}^{s} \alpha(t)dt < 0$ where α(t) =I $\otimes[\mu_1(t) - \mu_2(t)]$

Then (3.1) and (3.2) are unique solutions in M of the equations

$$u^1(t) = I \otimes \{(\mu_1(t) - \mu_2(t))v(t) + 1\}$$

$$v^1(t) = I \otimes \{(\mu_2(t) - \mu_1(t))v(t) - 1\}$$

respectively

If α(t) = $I \otimes [\mu_1(t) - \mu_2(t)]$ is just a bounded function and

$$\lim_{(t-s)} \sup \frac{1}{(t-s)}\int_s^t \alpha(T)dT < 0$$

When G $\in P_w$, Z is regular if and only if the only solution of (2.2) which belongs to M is the solution φ = 0.

The following result shows when a (μ_1, μ_2) – dichotomy cannot be destroyed by small perturbations in L^∞.

Theorem **:** Let us assume that system (2.2) has a (μ_1, μ_2) – dichotomy and that (3.1) and (3.2) are bounded functions on R. If H(t) $\in$ M and δ = sup $\{|H(t)| : t \in R\}$ then the

Perturbed equation

$$\varphi^1 = [G(t) + H(t)]\varphi, \quad t \in R$$

has a (μ^1_1, μ^1_2) – dichotomy for δ small enough where $\mu^1_1 = \mu_1 + 6L^3\delta$ and
$\mu^1_2 = \mu_2 - 6L^3\delta$.

Proof:- Since (2.2) admits a (μ_1, μ_2) – dichotomy (2.5) holds. Then by Lemma 2 [[2],p. 40],there exists a continuously differentiable invertible n^2xn^2 matrix τ(t) with $\|\tau(t)\| \le \sqrt{2}$, $\|\tau^{-1}(t)\| \le \sqrt{2}L$ such that the change of variables, φ=τ(t)Z transforms (2.3) into the system

$$Z^1 = [R(t) + S(t)]Z \quad (3.3)$$

whose coefficient matrix $R(t) = \tau^{-1}(t)G(t)\tau(t) - \tau^{-1}(t)\tau(t)$ is Hermitian commutes with P_1 and P_2, and satisfies $\|R(t)\| \le \|G(t)\|, \forall t \in R$ Moreover $\|S(t)\| = \|\tau^{-1}(t)H(t)\tau(t)\| \le 2L\delta, \forall t \in R$

Our next object is to transform the system (3.3) into a kinematically similar system whose coefficient matrix commutes with P_1 and P_2.

Denoting by $V(t) = (W \otimes Y)(t)$ the fundamental matrix of (2.2) and observing that $Z(t) = \tau^{-1}(t)V(t)$ is a fundamental matrix of the system $Z^1 = R(t)Z$, of order $n^2 \times n^2$, we obtain that

$$\|Z(t)P_iZ^{-1}(s)\| \le 2L^2 \exp\int_s^t \mu_i(T)dT \quad (t-s)(-1)^{i-1} \ge 0,\ i = 1,2 \quad (3.4)$$

Let us define an operator 'β' on the space $M(R, R^{n^2xn^2})$ of the continuous and bounded matrix functions as follows:

$$\beta H(t) = \int_{-\infty}^{t} Z(t)P_1Z^{-1}(s)[I_{n^2xn^2} - H(s)]S(s)[I_{n^2xn^2} + S(s)]Z(s)P_2Z^{-1}(t)ds$$

$$-\int_{t}^{\infty} Z(t)P_2Z^{-1}(s)[I_{n^2xn^2} - H(s)]S(s)[I_{n^2xn^2} + S(s)]Z(s)P_1Z^{-1}(t)ds$$

Using (3.4) and the fact that u^* and v^* are bounded on R^{n^2}, it can be proved that 'β' is a contraction and maps the ball $\|H\| \le 1/2$ into itself, if 'δ' is sufficiently small ,let H^* be the unique fixed point of β in $\|H\| \le 1/2$.

Making the transformation , $Z = [I_{n^2} + H^*(t)]V$ and arguing like [2] pp.42-44 then (3.3) becomes

$$V^1 = [R(t) + S^*(t)]V \quad (3.5)$$

where the coefficient matrices of (3.5) decomposes into two independent sub systems. The rest of the proof in similar manner to [2] pp 43-44.

Remark **:** In the case that $\mu_1 < 0 < \mu_2$ are constants i.e., the system (2.2) has an exponential dichotomy , the boundedness of u^* and v^* is automatically satisfied.

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